A topological state sum model for fermions on the circle

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Abstract

A simple state sum model for fermions on a 1-manifold is constructed. The model is independent of the triangulation and gives exactly the same partition function as the Dirac functional integral with zeta-function regularisation. Some implications for more realistic physical models are discussed.

1 Introduction

Discrete models are used in physics as a practical way of defining a functional integral. In lattice gauge theories one obtains a model that approximates the continuum theory better as the number of discrete lattice cells is increased. In the continuum limit, the gauge theory should of course regain the full rotational and translational symmetry.

Topological quantum field theories can be defined using discrete models called state sum models [1]. In this case the partition function is independent of the number of discrete cells (the simplexes of a triangulation). Similar models are used in quantum gravity, but the requirement that the partition function is independent of the triangulation is generally relaxed; however the independence must at least hold in a suitable limit in which the model approximates general relativity.

Realistic quantum gravity models should of course include matter fields and so it is important to investigate models with matter fields that are independent of the triangulation, either exactly, or in a limit. This is our motivation for studying state sum models with fermionic fields.

In this paper a very simple one-dimensional state sum model is presented in which a fermionic field is coupled to a background gauge field. A simple formula for the partition function of this model on a triangulated circle (i.e., a polygon) is presented in Section 2. It is demonstrated that the partition function is independent of the triangulation and depends only on the holonomy of the gauge field.

A heuristic argument is presented in Subsection 3.1 that shows that the state sum model has an action that is a discretisation of the continuum Dirac action for a massless fermion field coupled to the gauge connection. Then, in Subsection 3.2 and the two appendices, the partition function of this functional integral is calculated precisely to show that it is in fact exactly the same as the partition function of the state sum model.

The results are extended in Subsection 3.3 to models with a mass term. A state sum model with a discretised mass term is presented. However in this case the model does depend on the number of edges N in the triangulation of the circle. It is shown that as $N \to \infty$ it converges to a limit, so that the triangulation-independence is recovered in this limit. The limiting value is closely related to the Dirac functional integral with a mass term, differing only by a certain phase factor. Finally, the paper ends in Section 4 with a discussion of the results, including a comparison with operator cut-off regularisations of the Dirac functional integral.

The results presented here complement previous work constructing quantum gravity state sum models with fermion fields in dimension three [2, 3] and four [4, 5]. These works construct discrete analogues of the continuum Dirac functional integral according to the heuristic continuum limit, as considered here in Subsection 3.1, but do not have a direct comparison with the partition function of the continuum functional integral. The results presented here give the first precise comparison of a discrete fermionic model with the partition function of the continuum functional integral.

It is an interesting question as to whether our results can be generalised to a higher-dimensional model. Some related properties of the Dirac operator with a gauge field on a graph have been studied previously [6]; this suggests there may also be an extension of our state sum model to graphs.

2 The state sum model

2.1 Fermionic variables

In a functional integral, the fermion fields anticommute and so are elements of a Grassmann algebra. Therefore the state sum model is based on a finite-dimensional Grassmann algebra over $\mathbb{C}$. Such an algebra is determined by a number of generators $a_1, a_2, \ldots, a_l$ satisfying the relations

$$a_i a_j + a_j a_i = 0. \tag{2.1}$$

A function of the generators $f(a_1, \ldots, a_l)$ is a polynomial, which terminates at the highest monomial $a_l a_{l-1} \ldots a_1$ due to the anticommutation. The Berezin integral

$$\int \mathrm{d}a_1 \, \mathrm{d}a_2 \ldots \mathrm{d}a_l \; f \tag{2.2}$$

is defined to be the coefficient of $a_l a_{l-1} \ldots a_1$ in the expansion of f. No independent meaning is attached to the differentials in this formula, and they do not appear outside an integral. However the order of them in the integral is important; transposing two neighbouring differentials in the notation changes the sign of the integral.

It is possible to extend the above definitions to integration over a subset of coordinates, and perform the integral iteratively. So if $f = a_k a_{k-1} \ldots a_1 b$, where b is a polynomial in the remaining variables $a_{k+1}, \ldots, a_l$, then the integral is

$$\int \mathrm{d}a_1 \, \mathrm{d}a_2 \ldots \mathrm{d}a_k \; f = b, \tag{2.3}$$

with terms multiplying lower degree monomials in $a_1 \ldots a_k$ integrating to zero. An example of iteration is the formula

$$\int \mathrm{d}a_1 \, \mathrm{d}a_2 \; f = \int \mathrm{d}a_1 \left(\int \mathrm{d}a_2 \; f \right). \tag{2.4}$$

The integration enjoys translational invariance: if c is any polynomial in the generators excluding a_1, then

$$\int f(a_1 + c) \, \mathrm{d}a_1 = \int f(a_1) \, \mathrm{d}a_1. \tag{2.5}$$

In the applications of interest here the generators of the Grassmann algebra occur in pairs a_i, b_i and are also grouped into a number of n-dimensional vectors, such as

$$\psi = (a_1, a_2, \dots a_n), \quad \overline{\psi} = (b_1, b_2, \dots, b_n). \tag{2.6}$$

In this case the integral is defined with the notation

$$\int \mathrm{d}\psi \, \mathrm{d}\overline{\psi} = \int \mathrm{d}a_1 \, \mathrm{d}b_1 \, \mathrm{d}a_2 \, \mathrm{d}b_2 \dots \mathrm{d}a_n \, \mathrm{d}b_n.$$

Let M be an $n \times n$ matrix with entries in $\mathbb{C}$. The basic gaussian integral is

$$\int \mathrm{d}\psi \, \mathrm{d}\overline{\psi} \, e^{\overline{\psi} M \psi} = \det M, \tag{2.7}$$

which is proved by expanding the exponential.

The result (2.7) can be extended to the case of fermionic source terms. Take $\overline{c}, d$ to be n-component vectors with Grassmann-valued entries that are polynomial of odd degree in the remaining generators (i.e., excluding components of both ψ and $\overline{\psi}$), and M now an invertible matrix. We then have

$$\int \mathrm{d}\psi \, \mathrm{d}\overline{\psi} \, e^{\overline{\psi} M \psi + \overline{c}\psi + \overline{\psi} d} = \det M \, e^{-\overline{c} M^{-1} d}, \tag{2.8}$$

which is proved by first completing the square with the translations

$$\overline{\psi} \mapsto \overline{\psi} - \overline{c} M^{-1}, \tag{2.9a}$$
$$\psi \mapsto \psi - M^{-1} d, \tag{2.9b}$$

and then using (2.7).

2.2 Definition of the state sum model

The construction of the fermionic state sum model is based on fermionic variables at a discrete set of points. For this, a number of n-dimensional vectors ψ_i, $\overline{\psi}_i$ are used (with i labelling the vectors, not the components). The Grassmann algebra is the one generated by all of the components of all of the vectors.

The basic construction is the partition function for the interval $[0, 1] \subset \mathbb{R}$ decorated with a fixed $n \times n$ matrix Q, as depicted in Figure 1. The partition

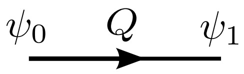

Figure 1: The partition function of an edge. The arrow indicates the orientation of the edge.

function then is

$$\mathbb{Z}^{Q}_{[0,1]} = e^{-\overline{\psi}_0 Q \psi_1}. \tag{2.10}$$

This has fermionic variables $\overline{\psi}_0$, associated to the boundary point 0, and ψ_1 associated to 1.

Gluing two such partition functions together is carried out using the following proposition, which states that one can multiply matrices by the use of Berezin integration.

Proposition.

$$\int \mathrm{d}\psi_1 \mathrm{d}\overline{\psi}_1 e^{-\overline{\psi}_0 Q_1 \psi_1} e^{\overline{\psi}_1 \psi_1} e^{-\overline{\psi}_1 Q_2 \psi_2} = e^{-\overline{\psi}_0 Q_1 Q_2 \psi_2}. \tag{2.11}$$

Proof. We have

$$\int \mathrm{d}\psi_1 \mathrm{d}\overline{\psi}_1\, e^{-\overline{\psi}_0 Q_1 \psi_1} e^{\overline{\psi}_1 \psi_1} e^{-\overline{\psi}_1 Q_2 \psi_2} = \int \mathrm{d}\psi_1 \mathrm{d}\overline{\psi}_1 e^{\overline{\psi}_1 I \psi_1 - (\overline{\psi}_0 Q_1)\psi_1 - \overline{\psi}_1 (Q_2 \psi_2)}, \tag{2.12}$$

where I is the $n \times n$ identity matrix. The result (2.11) follows using (2.8) with

$$\overline{c} = -\overline{\psi}_0 Q_1, \quad d = -Q_2 \psi_2, \quad M = I. \tag{2.13}$$

Our interpretation of formula (2.11) is that there is a bilinear form on the fermionic states,

$$(f, g) = \int \mathrm{d}\psi_1 \mathrm{d}\overline{\psi}_1\, f(\psi_1)\, e^{\overline{\psi}_1 \psi_1} g(\overline{\psi}_1), \tag{2.14}$$

and using this bilinear form to glue the partition functions results in

$$\mathbb{Z}^{Q_1}_{[0,1]} \mathbb{Z}^{Q_2}_{[0,1]} = \mathbb{Z}^{Q}_{[0,1]}, \tag{2.15}$$

with $Q = Q_1 Q_2$.

Clearly this can be iterated for the multiplication of any finite number of matrices, yielding the definition of the state sum model. Explicitly,

$$\int \mathrm{d}\psi_1\, \mathrm{d}\overline{\psi}_1 \ldots \mathrm{d}\psi_{N-1}\, \mathrm{d}\overline{\psi}_{N-1}\, e^{-\overline{\psi}_0 Q_1 \psi_1} e^{\overline{\psi}_1 \psi_1} e^{-\overline{\psi}_1 Q_2 \psi_2} \ldots e^{-\overline{\psi}_{N-1} Q_N \psi_N}$$
$$= e^{-\overline{\psi}_0 Q \psi_N} = \mathbb{Z}^{Q}_{[0,1]} \tag{2.16}$$

with $Q = Q_1 Q_2 \ldots Q_N$.

The left-most expression in (2.16) is to be interpreted as the definition of the fermionic state sum model on an oriented interval that is triangulated using N 1-simplexes. The equation itself is a statement that the partition function is independent of the triangulation, though of course provided the matrices Q and the orientation match. Geometrically, if the Q_i are all invertible, they may be viewed as the parallel transport operators for a connection 1-form on the interval. Then the partition function can be viewed as a function of the connection that is invariant under gauge transformations in the interior of the interval.

The state sum model on the left-hand side of (2.16) can be modified to include observables, that is, non-trivial functions of the intermediate variables ψ_i, $\overline{\psi}_i$. In this sense the model is richer than the evaluation of the partition function on the right-hand side. The physical content of the model will be investigated further in Section 3 by comparing it to a functional integral.

The partition function for the circle can be computed by gluing together the endpoints of the interval, as depicted in Figure 2.

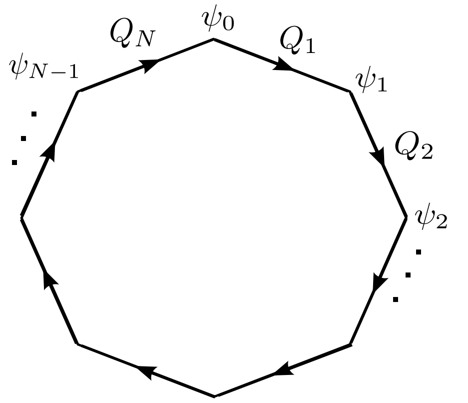

Figure 2: The state sum model on a triangulated circle.

Mathematically, this is done by including an extra factor $e^{\overline{\psi}_N \psi_N}$ in the integrand, identifying $\psi_0 = \psi_N$, $\overline{\psi}_0 = \overline{\psi}_N$, and integrating over the newly

introduced variables,

$$\mathbb{Z}^{Q}_{\mathrm{S}^1} = \int \mathrm{d}\psi_N \mathrm{d}\overline{\psi}_N \, e^{\overline{\psi}_N \psi_N} e^{-\overline{\psi}_N Q \psi_N} = \det{(I - Q)}\,, \tag{2.17}$$

where the last equality follows from (2.7) and the observation that $\overline{\psi}_N \psi_N$ and $\overline{\psi}_N Q \psi_N$ commute.

An immediate consequence of (2.17) is that $\mathbb{Z}^{Q}_{\mathrm{S}^1}$ vanishes if Q has an eigenvalue equal to 1, as is for example the case for the gauge group $\mathrm{S}O(2n+1)$. For $SO(2n)$, $\mathbb{Z}^{Q}_{\mathrm{S}^1}$ is real-valued, as the eigenvalues occur in complex conjugate pairs. There are however gauge groups for which $\mathbb{Z}^{Q}_{\mathrm{S}^1}$ is complex-valued. An example of particular interest is $\mathrm{U}(1)$, for which we have $Q = e^{-i\theta}$ with $\theta \in \mathbb{R}$, and so

$$\mathrm{U}(1): \quad \mathbb{Z}^{Q}_{\mathrm{S}^1} = 1 - e^{-i\theta}. \tag{2.18}$$

This example shows that the partition function depends on the orientation of the circle for some gauge groups. Given a circle with a connection, if the holonomy around one direction is the matrix Q, the holonomy around the opposite direction is Q^{-1}. For $\mathrm{S}O(n)$, $\det{(I-Q)} = \det{(I-Q^{-1})}$ for all Q, and so the partition function is independent of orientation. For $\mathrm{U}(n)$, however $\det{(I-Q)}$ is the complex conjugate of $\det{(I-Q^{-1})}$, and the $\mathrm{U}(1)$ example (2.18) shows that the partition function is sensitive to the orientation for generic Q.

2.3 Gauge transformations

Gauge transformations act on the partition function of $[0,1]$ by a linear transformation acting on each set of fermionic variables. Thus if U_0, U_1 are invertible matrices then the transformation is $\psi_1' = U_1 \psi_1$ and $\overline{\psi}_0' = (U_0)^{-1T} \overline{\psi}_0$, using the inverse transpose matrix. The matrix Q transforms as $Q' = U_0 Q U_1^{-1}$. It is clear then that the partition function for the interval (2.10) is invariant under gauge transformations,

$$Z^{Q'}\big(\overline{\psi}_0', \psi_1'\big) = Z^{Q}\big(\overline{\psi}_0, \psi_1\big).$$

Similarly, the partition function of the circle (2.17) is gauge-invariant.

Nothing so far has suggested that a variable ψ_i is related to $\overline{\psi}_i$. However if, as the notation suggests, one regards these as complex conjugate variables, then the corresponding transformations should be complex conjugate matrices, $(U_i)^{-1T} = \overline{U}_i$, which means that these matrices are unitary.

3 Comparison with the functional integral

The partition function for the circle with a connection is compared to the corresponding functional integral in this section. First it is shown, in a heuristic fashion, that increasing the number of 1-simplexes can be interpreted as the state sum integrand converging to the exponential of the Dirac action for a fermion field. Then a precise evaluation of the Dirac functional integral is compared to the state sum model.

3.1 Continuum limit of the action

This subsection demonstrates that the state sum model is a discrete version of the Dirac functional integral. This is done by looking at the action for the state sum model and taking the limit $N \to \infty$, assuming that the discrete values of the field sample values of a differentiable field $\psi(t)$. The result of this heuristic limiting process is the continuum Dirac action.

The state sum model for the partition function on a circle triangulated with N edges is

$$\mathbb{Z}^{Q}_{\mathrm{S}^1} = \int \left(\prod_{j=1}^{N} \mathrm{d}\psi_j \, \mathrm{d}\overline{\psi}_j \right) e^{-\overline{\psi}_N Q_1 \psi_1} e^{\overline{\psi}_1 \psi_1} ... e^{\overline{\psi}_{N-1} \psi_{N-1}} e^{-\overline{\psi}_{N-1} Q_N \psi_N} e^{\overline{\psi}_N \psi_N}. \quad (3.1)$$

The integrand can be rewritten as

$$\begin{aligned} J &= \left(e^{\overline{\psi}_N \psi_N} e^{-\overline{\psi}_N Q_1 \psi_1} \right) \left(e^{\overline{\psi}_1 \psi_1} e^{-\overline{\psi}_1 Q_2 \psi_2} \right) \dots \left(e^{\overline{\psi}_{N-1} \psi_{N-1}} e^{-\overline{\psi}_{N-1} Q_N \psi_N} \right) \\ &= e^{i \Delta t \sum_{j=0}^{N-1} \overline{\psi}_j i \left(\frac{Q_{j+1} \psi_{j+1} - \psi_j}{\Delta t} \right)}, \end{aligned} \quad (3.2)$$

with $\psi_N = \psi_0$.

With the choice $\Delta t = l/N$, where l is a positive constant, this looks precisely like a discretisation of the Dirac action on a circle of circumference l. Let t be the coordinate on the circle with range 0 to l. Then the j-th vertex corresponds to the coordinate $t = jl/N$, and Δt is the distance between neighbouring vertices.

The limit $N \to \infty$ is the limit $\Delta t \to 0$. In this limit the parallel transport of the connection is $Q_j = 1 - iA^a(t) X_a \Delta t + O(\Delta t)^2$, where iX_a are the generators of the relevant Lie algebra and $A^a(t)$ are the components of the connection 1-form.

Assuming the field values are differentiable, we have

$$\lim_{\Delta t \to 0} i \left(\frac{Q_{i+1} \psi_{i+1} - \psi_i}{\Delta t} \right) = \not{D} \psi(t), \quad (3.3)$$

where the gauge covariant Dirac operator $\not{D}$ is given by

$$\not{D} = i\frac{d}{dt} + A^a(t)X_a. \tag{3.4}$$

The single gamma matrix is equal to the complex number i. There is no spin connection contribution to (3.4) because the tangent space is one-dimensional and the Lie algebra $\mathfrak{so}(1)$ is trivial.

In the limit $\Delta t \to 0$ the sum converges to an integral,

$$\lim_{\Delta t \to 0} \Delta t \sum_{j=0}^{N-1} f(j\Delta t) = \int_0^l f\,\mathrm{d}t, \tag{3.5}$$

giving the continuum limit of the state-sum integrand as

$$\lim_{\Delta t \to 0} J \;=\; e^{i\int_0^l \mathrm{d}t\, \overline{\psi}(t)\not{D}\psi(t)}. \tag{3.6}$$

3.2 The functional integral

In this subsection the above heuristic considerations are confirmed by an explicit and precise evaluation of the functional integral in the U(n) case.

The continuum partition function is

$$\mathbb{F}^A_{S^1} = \int \mathcal{D}\psi\mathcal{D}\overline{\psi}\, e^{i\int_0^l \mathrm{d}t\, \overline{\psi}(t)\not{D}\psi(t)}, \tag{3.7}$$

where $\not{D}$ is given by (3.4) and the fermions obey periodic boundary conditions. These boundary conditions can be regarded as the choice of the 'Lie group' spin structure for the circle [14]. The functional integral in (3.7) is defined via zeta-function regularisation [7, 8], as reviewed in Appendix A. The result is

$$\mathbb{F}^A_{S^1} = \det(i\not{D}) = e^{i\frac{\pi}{2}\eta_{\not{D}}(0)} e^{-\frac{1}{2}\zeta'_{\not{D}^2}(0)}, \tag{3.8}$$

where the functions $\eta_{\not{D}}$ and $\zeta_{\not{D}^2}$ are defined in Appendix A.

It is shown in Appendix B that for U(1)

$$\mathbb{F}^A_{S^1} = \mathbb{Z}^Q_{\mathrm{S}^1}, \tag{3.9}$$

where Q is the holonomy of the connection A. The result (3.9) generalises immediately to U(n) by diagonalising the connection with a gauge transformation, whereupon the functional integral is the product of a number of U(1) functional integrals. The Dirac functional integral is invariant under

these gauge transformations because it depends only on the eigenvalues of the Dirac operator, which are gauge invariant.

The result (3.9) is surprising because the eigenvalues of the Dirac operator are unbounded and so the naïve determinant of the Dirac operator, the product of its eigenvalues, diverges. Somehow the discrete model both approximates the eigenvalues of the continuum operator yet also avoids the divergence, and miraculously imitates the zeta-function regularisation.

Some insight into the result (3.9) can be gained by comparing the eigenvalues for the continuum Dirac operator with the eigenvalues for its discrete version.

The discrete version of the Dirac operator is a matrix M acting on the vectors $\psi = \bigoplus_{j=1}^{N} \psi_j$. It is determined by

$$\overline{\psi} iM\psi = \sum_{j=0}^{N-1} \overline{\psi}_j \left(\psi_j - Q_{j+1}\psi_{j+1}\right), \tag{3.10}$$

and can be written in block form as

$$iM = \begin{pmatrix} 1 & -Q_2 & & & \\ & 1 & -Q_3 & & \\ & & \ddots & \ddots & \\ & & & 1 & -Q_N \\ -Q_1 & & & & 1 \end{pmatrix}. \tag{3.11}$$

where each Q_i is in $\mathrm{U}(n)$. Since we have seen that the partition function associated with M is exactly the same as the continuum partition function, it must be that the matrix M is in some sense approximating the differential operator $\not{D}$. We now make this more explicit.

For simplicity, we specialise to $\mathrm{U}(1)$. The eigenvalues of iM are $\mu_k = 1 - \alpha_k$, where α_k are the N roots of

$$Q = \alpha^N, \tag{3.12}$$

and the corresponding eigenvectors are

$$\begin{pmatrix} Q_1^{-1} \\ \alpha_k Q_2^{-1} Q_1^{-1} \\ \alpha_k^2 Q_3^{-1} Q_2^{-1} Q_1^{-1} \\ \cdots \\ \alpha_k^{N-1} Q_N^{-1} \ldots Q_1^{-1} \end{pmatrix}. \tag{3.13}$$

Taking $Q = e^{-i\theta}$ with $\theta \in [0, 2\pi)$, as in (2.18), we have

$$\mu_k = 1 - e^{-i\left(\frac{\theta+2k\pi}{N}\right)}, \tag{3.14}$$

where the distinct eigenvalues are obtained by selecting a suitable set of distinct values of k. For example, this can be done explicitly by taking $k = [(1-N)/2], \ldots, [(N-1)/2]$, where $[x]$ stands for the largest integer that is less than or equal to x.

To compare with the continuum Dirac operator, we note that the eigenvalues (3.14) of iM have the large N expansion

$$iM: \quad \mu_k = 1 - e^{-i\left(\frac{\theta+2\pi k}{N}\right)} = i\left(\frac{\theta+2\pi k}{N}\right) + O\left(\left(\frac{\theta+2\pi k}{N}\right)^2\right), \tag{3.15}$$

while in Appendix B it is shown that the eigenvalues of $i\not{D}$ are

$$i\not{D}: \quad \mu_k = i\left(\frac{\theta+2\pi k}{l}\right), \quad k \in \mathbb{Z}. \tag{3.16}$$

The expressions (3.15) and (3.16) coincide modulo $O(N^{-2})$ if the circle has length $l = N$ and k is held fixed. The matrix iM hence approximates the operator $i\not{D}$ in the sense that the eigenvalues of small modulus coincide in the limit of a large circle.

Note from (3.15) that the eigenvalues of M are complex but the imaginary part is subdominant as $N \to \infty$ with fixed k. This raises the question as to whether it is fruitful to think of M as a cut-off version of $\not{D}$. We shall return to this question in Section 4.

3.3 Mass term

In this subsection we show that inclusion of a mass term breaks triangulation independence, but gives a well-defined partition function as $N \to \infty$. We start now with a massive continuum fermionic action

$$S = \int_0^l \mathrm{d}t\, \overline{\psi}(t)\left(\not{D} - m\right)\psi(t), \tag{3.17}$$

and we follow the usual discretisation procedure using (3.3) and (3.5). The state sum is given by

$$\mathbb{I}_{\mathrm{S}^1} = \int \left(\prod_{j=0}^{N-1} d\psi_j d\overline{\psi}_j\right) e^{\sum_{j=0}^{N-1} \overline{\psi}_j((1-im\Delta t)\psi_j - Q_{j+1}\psi_{i+1})}, \tag{3.18}$$

where $\Delta t = l/N$, $\psi_N = \psi_0$, and the mass term has been discretised according to

$$\int_0^l m\overline{\psi}\psi \, \mathrm{d}t \to m\Delta t \sum_{j=0}^{N-1} \overline{\psi}_j \psi_j. \tag{3.19}$$

We can evaluate $\mathbb{I}_{\mathrm{S}^1}$ in a similar way as before to obtain

$$\mathbb{I}_{\mathrm{S}^1} = \det\left((1 - im\Delta t)^N - \prod_{i=1}^{N} Q_i \right), \tag{3.20}$$

which is clearly not triangulation independent. The triangulation dependence appears to have crept in with the introduction of a fixed mass scale m. A way to regain triangulation independence might be to relegate m from a fixed parameter to a function of N and a new fixed positive parameter m' by $1 - im\Delta t = e^{-im'l/N}$; however, this would mean allowing m to be complex-valued.

With m fixed, the limit $N \to \infty$ does however yield a well-defined answer,

$$\lim_{N\to\infty} \det\left((1 - im\Delta t)^N - \prod_{i=1}^{N} Q_i \right) = \det(e^{-iml} - Q), \tag{3.21}$$

where Q is the holonomy around the circle. This can be compared to the evaluation of the partition function with a mass term in the continuum $\mathrm{U}(n)$ case. The calculation is done identically but using $A' = A - m$. The answer comes out as

$$\mathbb{F}_{S^1}^{A'} = \det(1 - e^{2\pi i m} Q), \tag{3.22}$$

which differs from (3.21) by a phase factor.

The disparity is due to the different way in which the gauge field and mass term are treated. In the continuum calculation they are bundled together into one term. However, in the discrete calculation, the gauge field is realised as parallel transporters while the mass term couples fields at the same point.

4 Discussion

In this paper, a fermionic state sum model on one-dimensional manifolds with a connection is defined. The partition function of the state sum model on the circle is the same as the functional integral of a complex fermion field with the Dirac action and the usual coupling to the connection gauge field.

A curious feature of the partition function for the circle is that it does not depend on the length of the circle, whereas the eigenvalues of both the continuum Dirac operator $\not{D}$ and the discrete Dirac operator M clearly do.

Some more insight into this can be gained by examining an operator cut-off regularisation of the functional integral. Such a regularisation depends on a cut-off scale c. It replaces iD with $f(iD)$, where the function $f(z)$ of a complex variable is the identity function $f(z) = z$ for $|z| \ll c$, but $f(z) = 1$ for $|z| \gg c$. Thus it effectively removes the eigenvalues of D with magnitude above the cut-off c. An example of such a cut-off regularisation is the Schwinger proper time regularisation.

As $c \to \infty$, the regularised determinant diverges. A calculation for the Schwinger proper time regularisation shows that the leading asymptotic term for $\log\det|\not{D}|$ is $(l/\pi)\, c \log c$. This can be confirmed in a simple way for special cases, e.g. $A = 1/2$, using a sharp cut-off and Stirling's formula for the asymptotic expansion of the factorial.

The cut-off regularisation only agrees with the zeta function method once the leading asymptotic terms are removed [7]. It is worth noting that these leading divergent terms are proportional to l, thus explaining the role played by the length of the circle in evaluating this determinant. Thus to get a partition function that converges as $c \to \infty$, one can multiply the cut-off regularised determinant with a 'cosmological term' $e^{\Lambda(c)l}$, choosing a suitable function $\Lambda(c)$ to renormalise the coefficient of l as $c \to \infty$. This is the same as adding a term $\Lambda(c)l$ to the exponent in (3.7) when using an operator cut-off regularisation. The functional integral formula for this regularisation is then

$$\int \mathcal{D}\psi\mathcal{D}\overline{\psi}\; e^{i\int_0^l \mathrm{d}t\, (\overline{\psi}(t)\not{D}\psi(t) - i\Lambda)}. \tag{4.1}$$

Similar divergences do not occur with cut-off regularisations of the eta invariant in the phase term [9], so there are no additional parameters to renormalise for the phase of the partition function.

Our conclusion from this discussion is that the state sum model is a more subtle regularisation of the determinant of the Dirac operator than a mere operator cut-off.

Another aspect of this is the absence of the phenomenon of fermion doubling which commonly occurs with lattice discretisations of the Dirac action [10]. One of the standard assumptions of discussions of fermion doubling is that the action of the lattice theory is real (i.e. equal to its conjugate with a suitable definition of complex conjugation for fermionic variables). However the action used in the state sum model (2.16) is in fact not real, and the resulting complex nature of the eigenvalues of M appears to be a crucial aspect of the model.

Finally, it is worth making a brief mention of possible generalisations. One can include an integration over the U(n) gauge field in the state sum model using the normalised Haar measure on the group

$$\mathbb{T}_{[0,1]} = \int \mathrm{d}Q\ \mathbb{Z}^Q_{[0,1]}. \tag{4.2}$$

The resulting partition function $\mathbb{T}_{[0,1]}$ is independent of any background structure; (2.15) yields

$$\mathbb{T}_{[0,1]}\mathbb{T}_{[0,1]} = \mathbb{T}_{[0,1]} \tag{4.3}$$

so that $\mathbb{T}_{[0,1]}$ is a projection operator. The corresponding partition function for a circle is thus an integer, and can be interpreted in terms of counting states in the space of polynomials in the fermion variables at a point that are invariant under the group action.

Some aspects of the model presented here have been studied for the Dirac operator on a graph; the formula (3.21) is studied in [15]. The most interesting generalisation would be to models in higher dimensions. Whilst one cannot expect such explicit evaluations as presented here for the circle, we hope that the one-dimensional model clarifies some of the issues that may arise in higher dimensions.

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A Appendix: Zeta-function regularisation of the Dirac determinant

In this appendix, the zeta-function regularisation of $\det D$ for a self-adjoint operator D that is not necessarily positive definite [7] is reviewed. This regularisation is adapted to $\det(iD)$, which is compared with $\det D$.

A.1 Positive definite D

To begin, suppose D is a Hermitian, strictly positive operator in a finite-dimensional Hilbert space. The zeta-function $\zeta_D(s)$ of D is defined for $s \in \mathbb{C}$

by

$$\zeta_D(s) = \sum_k \frac{1}{\lambda_k^s}, \tag{A.1}$$

where λ_k are the eigenvalues of D. As the eigenvalues are finitely many and positive, $\zeta_D(s)$ is well-defined and holomorphic in s. An elementary computation shows that

$$\det D = \prod_k \lambda_k = e^{-\zeta_D'(0)}. \tag{A.2}$$

The point of (A.2) is that the rightmost expression can be adopted as the definition of $\det D$ even when the Hilbert space is infinite dimensional, provided the spectrum of D is still discrete and the sum in (A.1) converges for sufficiently large $\mathrm{Re}\, s$ to define $\zeta_D(s)$ as a function that can be analytically continued to $s = 0$. The analytic continuation in s provides a prescription that removes the infinity from the formally divergent product $\prod_k \lambda_k$.

A.2 Indefinite D

Suppose that D is an indefinite Hermitian operator in a finite-dimensional Hilbert space, such that the spectrum of D does not contain zero. We wish to express $\det D$ and $\det(iD)$ in a form similar to (A.2). The new issue is to accommodate the negative and imaginary eigenvalues.

Let λ_k denote the eigenvalues of D, enumerated so that $\lambda_k > 0$ for $k > 0$ and $\lambda_k < 0$ for $k \leq 0$. We define two zeta-functions by

$$\zeta_{D,\epsilon}(s) = \sum_{k>0} \frac{1}{\lambda_k^s} + e^{i\epsilon\pi s} \sum_{k\leq 0} \frac{1}{(-\lambda_k)^s}, \tag{A.3a}$$

$$\zeta_{iD}(s) = e^{-i\pi s/2} \sum_{k>0} \frac{1}{\lambda_k^s} + e^{i\pi s/2} \sum_{k\leq 0} \frac{1}{(-\lambda_k)^s}, \tag{A.3b}$$

where $\epsilon \in \{1, -1\}$. For integer argument these functions agree with naïvely allowing negative or imaginary eigenvalues in (A.1).

We next define the eta-function of D by

$$\eta_D(s) = \sum_k \frac{\operatorname{sgn} \lambda_k}{|\lambda_k|^s} = \sum_{k>0} \frac{1}{\lambda_k^s} - \sum_{k\leq 0} \frac{1}{(-\lambda_k)^s}. \tag{A.4}$$

Finally, since D^2 is positive definite, its zeta-function is defined by the replacements $D \to D^2$ and $\lambda_k \to \lambda_k^2$ in (A.1). It follows that

$$\zeta_{D^2}(s/2) = \sum_k \frac{1}{|\lambda_k|^s} = \sum_{k>0} \frac{1}{\lambda_k^s} + \sum_{k\leq 0} \frac{1}{(-\lambda_k)^s}. \tag{A.5}$$

All the functions in (A.3), (A.4) and (A.5) are well-defined and holomorphic. They satisfy

$$\zeta_{D,\epsilon}(s) = \frac{1}{2}\bigl(1 + e^{i\epsilon\pi s}\bigr)\,\zeta_{D^2}(s/2) + \frac{1}{2}\bigl(1 - e^{i\epsilon\pi s}\bigr)\,\eta_D(s), \tag{A.6a}$$

$$\zeta_{iD}(s) = \cos(\pi s/2)\zeta_{D^2}(s/2) - i\sin(\pi s/2)\eta_D(s), \tag{A.6b}$$

and differentiation of (A.6) at $s = 0$ yields

$$\zeta'_{D,\epsilon}(0) = \frac{1}{2}\zeta'_{D^2}(0) + \frac{i\epsilon\pi}{2}\bigl(\zeta_{D^2}(0) - \eta_D(0)\bigr), \tag{A.7a}$$

$$\zeta'_{iD}(0) = \frac{1}{2}\zeta'_{D^2}(0) - \frac{i\pi}{2}\eta_D(0). \tag{A.7b}$$

We are now ready to turn to the determinants. An elementary computation using (A.3) shows that

$$\det D = \prod_k \lambda_k = e^{-\zeta'_{D,\epsilon}(0)}, \tag{A.8a}$$

$$\det(iD) = \prod_k (i\lambda_k) = e^{-\zeta'_{iD}(0)}. \tag{A.8b}$$

Using (A.7), we thus have

$$\det D = e^{i\epsilon\frac{\pi}{2}\left(\eta_D(0) - \zeta_{D^2}(0)\right)}\, e^{-\frac{1}{2}\zeta'_{D^2}(0)}, \tag{A.9a}$$

$$\det(iD) = e^{i\frac{\pi}{2}\eta_D(0)}\, e^{-\frac{1}{2}\zeta'_{D^2}(0)}. \tag{A.9b}$$

Formulas (A.9) provide definitions for $\det D$ and $\det(iD)$ that extend to the case when the Hilbert space is infinite-dimensional and separable and the spectrum of D is discrete with suitable asymptotic properties. The values $\eta_D(s)$ and $\zeta_{D^2}(s/2)$ are in this situation defined by (A.4) and (A.5) for sufficiently large $\operatorname{Re} s$ and analytically continued to $s = 0$.

An important difference between $\det D$ and $\det(iD)$ arises from the phases in (A.3). In the definition of ζ_{iD} (A.3b), we chose the phases of the positive and negative eigenvalue terms to be the opposite for real argument, with the consequence that in the finite-dimensional case ζ_{iD} is real-valued for real argument whenever the spectrum of D is invariant under $D \to -D$. In the definition of $\zeta_{D,\epsilon}$ (A.3a), by contrast, the branch of $(-1)^{-s}$ in the negative eigenvalue terms cannot be fixed by a similar symmetry argument, and we parametrised this ambiguity by $\epsilon \in \{1, -1\}$, finding that ϵ still survives in the final formula (A.9a) for $\det D$. In the finite-dimensional case $\eta_D(0) - \zeta_{D^2}(0)$ is an even integer, and the right-hand side of (A.9a) is hence independent

of ϵ. In the infinite-dimensional case, however, the two values of ϵ can yield different regularised values for $\det D$: an example will occur in Appendix B. The choice $\epsilon = 1$ is related to our regularisation of $\det(iD)$ since $\zeta_{D,1}(s) = e^{i\pi s/2}\zeta_{iD}(s)$, whereas the formulas in [7] make the choice $\epsilon = -1$.

B Appendix: Dirac determinant on the circle for $\mathrm{U}(1)$

In this appendix we evaluate $\det(\not{D})$ and $\det(i\not{D})$ for the Dirac operator $\not{D}$ (3.4) with the gauge group U(1), using the regularisation (A.9).

For U(1), the Dirac operator (3.4) reduces to $\not{D} = i\frac{d}{dt} + A(t)$, where A is a real-valued function of the coordinate $t \in [0, l]$, the Hilbert space is $L^2\big([0, l]\big)$, and both A and the domain of $\not{D}$ have periodic boundary conditions. By a gauge transformation we may assume A to take a constant value, which we denote by $2\pi a/l$, and by a further gauge transformation we may assume $a \in [0, 1)$. The holonomy of A is $Q = e^{-2\pi i a}$. Note that a is uniquely determined by the holonomy.

Solving the eigenvalue equation $\not{D}\psi = \lambda\psi$, subject to the l-periodicity, shows that the eigenvalues are $\lambda_k = 2\pi(k+a)/l$ where $k \in \mathbb{Z}$. We exclude the special case $a = 0$, in which one eigenvalue vanishes. We then have $a \in (0, 1)$, all the eigenvalues are non-vanishing, and we are in the situation covered by Appendix A.

Consider first $\zeta_{\not{D}^2}$. Assuming $\mathrm{Re}\, s > 1$, we have

$$\begin{aligned} \zeta_{\not{D}^2}(s/2) &= (2\pi/l)^{-s} \sum_{k\in\mathbb{Z}} \frac{1}{|k+a|^s} \\ &= (2\pi/l)^{-s} \left(\sum_{j=0}^{\infty} \frac{1}{(j+a)^s} + \sum_{j=0}^{\infty} \frac{1}{(j+1-a)^s} \right) \\ &= (2\pi/l)^{-s} \big(\zeta_H(s, a) + \zeta_H(s, 1-a) \big), \end{aligned} \tag{B.1}$$

where the sums are absolutely convergent and the Hurwitz zeta function ζ_H is defined by [11]

$$\zeta_H(s, q) = \sum_{j=0}^{\infty} \frac{1}{(j+q)^s}. \tag{B.2}$$

Analytically continuing to $s = 0$ and using 25.11.13, 25.11.18 and 5.5.3

in [11], we find

$$\zeta_{\not{D}^2}(0) = 0, \tag{B.3a}$$

$$e^{-\frac{1}{2}\zeta'_{\not{D}^2}(0)} = 2\sin\pi a. \tag{B.3b}$$

Consider now $\eta_{\not{D}}$. Assuming again $\operatorname{Re} s > 1$, we have

$$\begin{aligned}\eta_{\not{D}}(s) &= (2\pi/l)^{-s}\sum_k \frac{\operatorname{sgn}(k+a)}{|k+a|^s} \\ &= (2\pi/l)^{-s}\big(\zeta_H(s,a) - \zeta_H(s,1-a)\big).\end{aligned} \tag{B.4}$$

Analytically continuing to $s = 0$ and using 25.11.13 in [11], we find

$$\eta_{\not{D}}(0) = 1 - 2a, \tag{B.5}$$

in agreement with [12, §1.13].

Using (A.9) with (B.3) and (B.5), and recalling $Q = e^{-2\pi i a}$, we have

$$\det \not{D} = \begin{cases} 1 - Q & \text{for } \epsilon = 1, \\ 1 - Q^{-1} & \text{for } \epsilon = -1, \end{cases} \tag{B.6a}$$

$$\det(i\not{D}) = 1 - Q. \tag{B.6b}$$

Note that $\det \not{D}$ and $\det(i\not{D})$ depend only on the holonomy and not on l. The reason for the l-independence can be traced to the property (B.3a).

The modulus of the final result (B.6b) for $\det(i\not{D})$ agrees with the calculation in the physics literature of the ratio of two such determinants with different values of a [13]. However the phase does not agree, presumably due to the fact that the definition of this ratio in [13] is given as an infinite product that is not absolutely convergent.

Finally, it is worth noting that when $a = 0$, the formulas B.6a and B.6b all extend by continuity to give determinant equal to 0. This is consistent with the fact that $\not{D}$ has 0 as an eigenvalue.

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